

2.11 Fourier series and transforms

Fourier series

Real form	$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \quad (2.474)$	$f(x)$ periodic function, period $2L$ a_n, b_n Fourier coefficients
	$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx \quad (2.475)$	
	$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx \quad (2.476)$	
Complex form	$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp \left(\frac{\mathrm{i} n \pi x}{L} \right) \quad (2.477)$	c_n complex Fourier coefficient
	$c_n = \frac{1}{2L} \int_{-L}^L f(x) \exp \left(\frac{-\mathrm{i} n \pi x}{L} \right) dx \quad (2.478)$	
Parseval's theorem	$\frac{1}{2L} \int_{-L}^L f(x) ^2 dx = \frac{a_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \quad (2.479)$	modulus
	$= \sum_{n=-\infty}^{\infty} c_n ^2 \quad (2.480)$	

Fourier transform^a

Definition 1	$F(s) = \int_{-\infty}^{\infty} f(x) e^{-2\pi \mathrm{i} x s} dx \quad (2.481)$	$f(x)$ function of x $F(s)$ Fourier transform of $f(x)$
	$f(x) = \int_{-\infty}^{\infty} F(s) e^{2\pi \mathrm{i} x s} ds \quad (2.482)$	
Definition 2	$F(s) = \int_{-\infty}^{\infty} f(x) e^{-\mathrm{i} x s} dx \quad (2.483)$	
	$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) e^{\mathrm{i} x s} ds \quad (2.484)$	
Definition 3	$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-\mathrm{i} x s} dx \quad (2.485)$	
	$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{\mathrm{i} x s} ds \quad (2.486)$	

^aAll three (and more) definitions are used, but definition 1 is probably the best.



Fourier transform theorems^a

Convolution	$f(x) * g(x) = \int_{-\infty}^{\infty} f(u)g(x-u) du$	(2.487)	f, g general functions * convolution
Convolution rules	$f * g = g * f$ $f * (g * h) = (f * g) * h$	(2.488) (2.489)	f $f(x) \rightleftharpoons F(s)$ g $g(x) \rightleftharpoons G(s)$
Convolution theorem	$f(x)g(x) \rightleftharpoons F(s) * G(s)$	(2.490)	$\rightleftharpoons$ Fourier transform relation
Autocorrelation	$f^*(x) \star f(x) = \int_{-\infty}^{\infty} f^*(u-x)f(u) du$	(2.491)	$\star$ correlation f^* complex conjugate of f
Wiener-Khintchine theorem	$f^*(x) \star f(x) \rightleftharpoons F(s) ^2$	(2.492)	
Cross-correlation	$f^*(x) \star g(x) = \int_{-\infty}^{\infty} f^*(u-x)g(u) du$	(2.493)	
Correlation theorem	$h(x) \star j(x) \rightleftharpoons H(s)J^*(s)$	(2.494)	h, j real functions H $H(s) \rightleftharpoons h(x)$ J $J(s) \rightleftharpoons j(x)$
Parseval's relation ^b	$\int_{-\infty}^{\infty} f(x)g^*(x) dx = \int_{-\infty}^{\infty} F(s)G^*(s) ds$	(2.495)	
Parseval's theorem ^c	$\int_{-\infty}^{\infty} f(x) ^2 dx = \int_{-\infty}^{\infty} F(s) ^2 ds$	(2.496)	
Derivatives	$\frac{df(x)}{dx} \rightleftharpoons 2\pi i s F(s)$ $\frac{d}{dx}[f(x) * g(x)] = \frac{df(x)}{dx} * g(x) = \frac{dg(x)}{dx} * f(x)$	(2.497) (2.498)	

^aDefining the Fourier transform as $F(s) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x s} dx$.

^bAlso called the "power theorem."

^cAlso called "Rayleigh's theorem."

Fourier symmetry relationships

$f(x)$	$\rightleftharpoons$	$F(s)$	definitions
even	$\rightleftharpoons$	even	real: $f(x) = f^*(x)$
odd	$\rightleftharpoons$	odd	imaginary: $f(x) = -f^*(x)$
real, even	$\rightleftharpoons$	real, even	even: $f(x) = f(-x)$
real, odd	$\rightleftharpoons$	imaginary, odd	odd: $f(x) = -f(-x)$
imaginary, even	$\rightleftharpoons$	imaginary, even	Hermitian: $f(x) = f^*(-x)$
complex, even	$\rightleftharpoons$	complex, even	anti-Hermitian: $f(x) = -f^*(-x)$
complex, odd	$\rightleftharpoons$	complex, odd	
real, asymmetric	$\rightleftharpoons$	complex, Hermitian	
imaginary, asymmetric	$\rightleftharpoons$	complex, anti-Hermitian	



Fourier transform pairs^a

$$f(x) \Rightarrow F(s) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i s x} dx \quad (2.499)$$

$$f(ax) \Rightarrow \frac{1}{|a|} F(s/a) \quad (a \neq 0, \text{ real}) \quad (2.500)$$

$$f(x-a) \Rightarrow e^{-2\pi i a s} F(s) \quad (a \text{ real}) \quad (2.501)$$

$$\frac{d^n}{dx^n} f(x) \Rightarrow (2\pi i s)^n F(s) \quad (2.502)$$

$$\delta(x) \Rightarrow 1 \quad (2.503)$$

$$\delta(x-a) \Rightarrow e^{-2\pi i a s} \quad (2.504)$$

$$e^{-a|x|} \Rightarrow \frac{2a}{a^2 + 4\pi^2 s^2} \quad (a > 0) \quad (2.505)$$

$$xe^{-a|x|} \Rightarrow \frac{8i\pi as}{(a^2 + 4\pi^2 s^2)^2} \quad (a > 0) \quad (2.506)$$

$$e^{-x^2/a^2} \Rightarrow a\sqrt{\pi}e^{-\pi^2 a^2 s^2} \quad (2.507)$$

$$\sin ax \Rightarrow \frac{1}{2i} \left[\delta\left(s - \frac{a}{2\pi}\right) - \delta\left(s + \frac{a}{2\pi}\right) \right] \quad (2.508)$$

$$\cos ax \Rightarrow \frac{1}{2} \left[\delta\left(s - \frac{a}{2\pi}\right) + \delta\left(s + \frac{a}{2\pi}\right) \right] \quad (2.509)$$

$$\sum_{m=-\infty}^{\infty} \delta(x-ma) \Rightarrow \frac{1}{a} \sum_{n=-\infty}^{\infty} \delta\left(s - \frac{n}{a}\right) \quad (2.510)$$

$$f(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases} \quad (\text{"step"}) \Rightarrow \frac{1}{2}\delta(s) - \frac{i}{2\pi s} \quad (2.511)$$

$$f(x) = \begin{cases} 1 & |x| \leq a \\ 0 & |x| > a \end{cases} \quad (\text{"top hat"}) \Rightarrow \frac{\sin 2\pi as}{\pi s} = 2a \operatorname{sinc} 2as \quad (2.512)$$

$$f(x) = \begin{cases} \left(1 - \frac{|x|}{a}\right) & |x| \leq a \\ 0 & |x| > a \end{cases} \quad (\text{"triangle"}) \Rightarrow \frac{1}{2\pi^2 a s^2} (1 - \cos 2\pi as) = a \operatorname{sinc}^2 as \quad (2.513)$$

^aEquation (2.499) defines the Fourier transform used for these pairs. Note that $\operatorname{sinc} x \equiv (\sin \pi x)/(\pi x)$.

